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Non-local higher-order symmetries for the Federbush model

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Abstract. By introduction of non-local variables four first-order non-local symmetries are obtained for the Federbush model. Moreover the Lie algebraic structure of the non-local symmetries is discussed.

1. Introduction

Symmetries play an important role in the construction of solutions of partial differential equations in mathematical physics. Moreover they provide us with information on properties of solutions. The Lie algebra of infinitesimal symmetries, vector fields which generate parameter groups of symmetries, provides us with useful tools to construct classes of special solutions of the partial differential equation, obtained by reduction using sub-Lie algebras. This method is exploited by Winternitz *et al* [1, 2].

The classical Noether theorem and its generalisations relate variational symmetries to conservation laws [3].

The concept of *generalised symmetry* generalises the classical notion of infinitesimal symmetry by requiring not only the invariance of the (system of) partial differential equations but the invariance of a partial differential equation together with all of its differential consequences. Generalised symmetries are sometimes called higher-order symmetries or Lie–Bäcklund transformations. Generalised symmetries flourish in the field of so-called soliton equations or integrable systems such as KdV, nonlinear Schrödinger and sine-Gordon equations, and allow the construction of multisoliton solutions and higher-order conservation laws. The notion of *non-local symmetry* based on taking into account integral as well as differential consequences of the partial differential equation has been described in an elegant way by Vinogradov and Krasilshchik [4], introducing the theory of coverings.

Recent work by Bluman *et al* [5] on this subject fits into this theory. Non-local symmetries led to the construction of the Cole Hopf transformation, linearising Burgers' equation. The famous Lenard recursion operator for generalised symmetries and higher-order conservation laws of KdV equation recursion operators and bi-Hamiltonian structures of soliton equations are obtained from non-local higher-order symmetries [6]. The pure technical mathematical computations which are extensive, especially in the computations of higher-order symmetries of partial differential equations, motivated researchers to construct computer algebra programs to carry out these computations on computer systems. Schwarz [7] constructed a program which

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deals with the determination of point symmetries of differential equations, a program which runs automatically in the symbolic language REDUCE.

One of us (PHMK) constructed a program based on a description of partial differential equations using differential forms, being developed in such a way as to be used *interactively*, which has advantages in avoiding expression well. The program offers the facilities to be used in the construction of generalised and non-local symmetries [8].

In this paper we investigate the existence of non-local higher-order symmetries of the Federbush model. The Federbush model describes the nonlinear interaction between two fermions and is given by a system of partial differential equations:

$$\begin{pmatrix} \mathrm{i}(\partial_t+\partial_x) & -m(s) \\ -m(s) & \mathrm{i}(\partial_t+\partial_x) \end{pmatrix}\begin{pmatrix} \psi_{s,1} \\ \psi_{s,2} \end{pmatrix} = 4s\pi\lambda\begin{pmatrix} |\psi_{-s,2}|^2\psi_{s,1} \\ |\psi_{-s,2}|^2\psi_{s,2} \end{pmatrix}$$

where $s=\pm 1$, $\psi_{s,1}$, $\psi_{s,2}$ are complex valued functions.

Infinitesimal symmetries are vector fields which leave the differential equation invariant; in terms of differential geometry [9] this amounts to

$$\mathscr{L}_V(I)\subset I \tag{1.1}$$

where I is a closed ideal of differential forms describing the partial equation, $\mathscr{L}_V$ denotes the Lie derivative with respect to the vector field V. In order to describe generalised symmetries, the notion of an infinite jet bundle [9] $J^\infty(M, N)$ of M and N has to be introduced, where M is the space of independent variables where local coordinates are $x_1^1, \ldots, x^m$, and N is the space of the dependent variables where local coordinates are given by $z^1, \ldots, z^n$. In this notion the independent, dependent variables and all partial derivatives are considered as independent quantities. The partial differential equation and its differential consequences are just algebraic equations on the infinite jet bundle $J^\infty(M, N)$, where coordinates are $x^1, \ldots, x^m, z^1, z^2, \ldots, z^n, z^1_{x_1,\ldots}$, $z^\mu_{x_1,\ldots,x_{ik}}, \ldots$.

Generalised symmetries are now described as formal vector fields on $J^\infty(M, N)$ which leave invariant the differential equation and all of its differential consequences. In terms of differential forms this amounts to

$$\mathscr{L}_V(D^\infty I)\subset D^\infty I \tag{1.2a}$$

whereas in (1.2) $D^\infty I$ is the infinite prolongation of the closed ideal I. Since the vector fields are supposed to be dependent only of a finite number of variables of $J^\infty(M, N)$ condition (1.2) reduces to [8]

$$\mathscr{L}_V(I)\subset D^r I. \tag{1.2b}$$

In a similar way we can allow non-local variables or potentials to enter as variables in the components of vector fields. Due to their nature equations for these potentials are added to the original differential equation. With these additional equations we can associate differential forms. Denoting

$$D^r(I; p_1, \ldots, p_k)$$

as the smallest differential ideal containing both $D^r(I)$ and the forms associated wiih $p_1, \ldots, p_k$ then we can generalise condition (1.2) to

$$\mathscr{L}_V(I)\subset D^r(I, p_1, \ldots, p_k) \tag{1.3}$$

leading to the concept of non-local symmetries V [4].

In section 2 we review some of the results obtained in [10] and [11]. In section 3 we derive four non-local higher symmetries while in section 4 the Lie algebraic structure is discussed.

2. Symmetries

The Federbush Model is described by

$$\begin{pmatrix} \mathrm{i}(\partial_t+\partial_x) & -m(s) \\ -m(s) & \mathrm{i}(\partial_t-\partial_x) \end{pmatrix}\begin{pmatrix} \psi_{s,1} \\ \psi_{s,2} \end{pmatrix} = 4s\pi\lambda\begin{pmatrix} |\psi_{-s,2}|^2\psi_{s,1} \\ |\psi_{-s,1}|^2\psi_{s,2} \end{pmatrix} \tag{2.1}$$

with $s=\pm 1$. $\psi_{s,1}$ and $\psi_{s,2}$ are analytic functions $\mathbb{R}^2\to\mathbb{C}$. Suppressing the factor 4π ($\lambda'=4\pi\lambda$) and introducing the eight variables u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4 by:

$$\begin{aligned} &\psi_{+1,1}=u_1+\mathrm{i}v_1 \qquad &&\psi_{+1,2}=u_2+\mathrm{i}v_2 \qquad &&m(+1)=m_1 \\ &\psi_{-1,1}=u_3+\mathrm{i}v_3 \qquad &&\psi_{-1,2}=u_4+\mathrm{i}v_4 \qquad &&m(-1)=m_2 \end{aligned} \tag{2.2}$$

equation (2.1) is rewritten as a sytem of eight nonlinear partial differential equations for the functions $u_1,\ldots,v_4$:

$$\begin{aligned} &u_{1t}+u_{1x}-m_1v_2=\lambda R_4v_1 \\ &-v_{1t}-v_{1x}-m_1u_2=\lambda R_4u_1 \\ &u_{2t}-u_{2x}-m_1v_1=-\lambda R_3v_2 \\ &-v_{2t}+v_{2x}-m_1u_1=-\lambda R_3u_2 \\ &u_{3t}+u_{3x}-m_2v_4=-\lambda R_2v_3 \\ &-v_{3t}-v_{3x}-m_2u_4=-\lambda R_2u_3 \\ &u_{4t}-u_{4x}-m_2v_3=\lambda R_1v_4 \\ &-v_{4t}+v_{4x}-m_2u_3=\lambda R_1u_4 \end{aligned} \tag{2.3}$$

with $R_i=u_i^2+v_i^2$ for $i=1,\ldots,4$.

In order to calculate the symmetries we introduce the differential ideal I, generated by the differential 1-forms:

$$\begin{aligned} &\alpha(1)=\mathrm{d}u_1-u_{1x}\,\mathrm{d}x-H(1,1)\,\mathrm{d}t \\ &\alpha(2)=\mathrm{d}v_1-v_{1x}\,\mathrm{d}x-H(1,2)\,\mathrm{d}t \\ &\alpha(3)=\mathrm{d}u_2-u_{2x}\,\mathrm{d}x-H(1,3)\,\mathrm{d}t \\ &\alpha(4)=\mathrm{d}v_2-v_{2x}\,\mathrm{d}x-H(1,4)\,\mathrm{d}t \\ &\alpha(5)=\mathrm{d}u_3-u_{3x}\,\mathrm{d}x-H(1,5)\,\mathrm{d}t \\ &\alpha(6)=\mathrm{d}v_3-v_{3x}\,\mathrm{d}x-H(1,6)\,\mathrm{d}t \\ &\alpha(7)=\mathrm{d}u_4-u_{4x}\,\mathrm{d}x-H(1,7)\,\mathrm{d}t \\ &\alpha(8)=\mathrm{d}v_4-v_{4x}\,\mathrm{d}x-H(1,8)\,\mathrm{d}t. \end{aligned} \tag{2.4}$$

The functions $H(1,1)$ to $H(1,8)$ are obtained by solving the equations (2.3) for $u_{1t},\ldots,v_{4t}$. The remaining variables x, t, $u_1,\ldots,v_4$, $u_{1x},\ldots,v_{4x}$ are now independent.

A useful property of the equations is that they allow a grading:

$$\begin{aligned}
&\deg(x)=\deg(t)=-2 && \deg(\partial_x)=\deg(\partial_t)=2\\
&\deg(u_1)=\ldots=\deg(v_4)=1 && \deg(\partial_{u_1})=\deg(\partial_{v_4})=-1\\
&\deg(m_1)=\deg(m_2)=2 && \\
&\mathrm{eeg}(u_{1x})=\ldots=\deg(v_{4x})=3 && \deg(\partial_{u_{1x}})=\deg(\partial_{v_{4x}})=-3.
\end{aligned} \qquad (2.5)$$

We can restrict our attention to look for homogeneous symmetries, because a vector field is a symmetry if and only if its homogenous parts are symmetries.

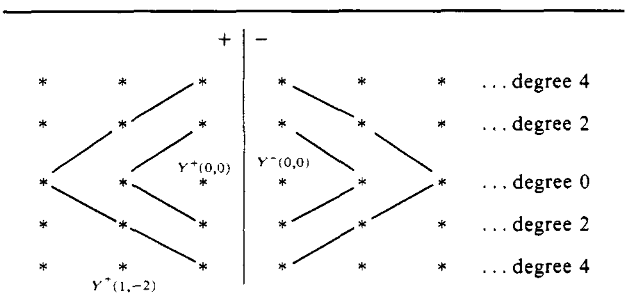
When looking for Lie-Bäcklund transformations the differential ideal $D^r(I)$ is generated by the differential 1-forms (2.4) and their Lie derivatives with respect to the total partial derivative vector fields D_x and D_t up to order r. For first-order Lie-Bäcklund transformations this means that $D^1(I)$ is generated by $\alpha(1)$ to $\alpha(8)$ and:

$$\begin{aligned}
\alpha(9)&=\mathrm{d}u_{1x}-u_{1xx}\,\mathrm{d}x-H(2,1)\,\mathrm{d}t\\
\alpha(10)&=\mathrm{d}v_{1x}-v_{1xx}\,\mathrm{d}x-H(2,2)\,\mathrm{d}t\\
\alpha(11)&=\mathrm{d}u_{2x}-u_{2xx}\,\mathrm{d}x-H(2,3)\,\mathrm{d}t\\
\alpha(12)&=\mathrm{d}v_{2x}-v_{2xx}\,\mathrm{d}x-H(2,4)\,\mathrm{d}t\\
\alpha(13)&=\mathrm{d}u_{3x}-u_{3xx}\,\mathrm{d}x-H(2,5)\,\mathrm{d}t\\
\alpha(14)&=\mathrm{d}v_{3x}-v_{3xx}\,\mathrm{d}x-H(2,6)\,\mathrm{d}t\\
\alpha(15)&=\mathrm{d}u_{4x}-u_{4xx}\,\mathrm{d}x-H(2,7)\,\mathrm{d}t\\
\alpha(16)&=\mathrm{d}v_{4x}-v_{4xx}\,\mathrm{d}x-H(2,8)\,\mathrm{d}t
\end{aligned} \qquad (2.6)$$

where $H(2,1)$ to $H(2,8)$ are obtained by differentiating $H(1,*)$ with respect to x.

The several Lie-Bäcklund transformations already found [10] form a direct sum of two commuting algebras (denoted + and −) and are reflected in the following table 1:

Table 1. Lie algebra of local symmetries.



Symmetries in a given row of table 1 have the same degree towards the grading. This degree is minimal for the row with the (two) point symmetries $Y^{\pm}(0,0)$ and increases in both directions by two degrees per row.

Each symmetry shown is a homogeneous polynomial in the variables x and t. In a column the several symmetries have the same degree with respect to the variables x and t. The colum $Y^{\pm}(i,*)$ contains the symmetries with degree i. In particular the columns $Y^{\pm}(0,*)$ are independent of x and t and the columns $Y^{\pm}(1,*)$ are linear in x and t.

Lie-Bäcklund transformations with the same order are connected by lines. The precise forms of these symmetries tend to be rather massive. At this moment we only mention:

$$
\begin{aligned}
Y^+(0,0) &= -v_1\partial_{u_1} + u_1\partial_{v_1} - v_2\partial_{u_2} + u_2\partial_{v_2}\\
Y^-(0,0) &= -v_3\partial_{u_3} + u_3\partial_{v_3} + v_4\partial_{u_4} + u_4\partial_{v_4}\\
Y^+(0,-1) &= \tfrac{1}{2}[-\lambda v_1(R_3+R_4) - m_1 v_2 + 2u_{1x}]\partial_{u_1} + \tfrac{1}{2}[\lambda u_1(R_3+R_4) + m_1 u_2 + 2v_{1x}]\partial_{v_1}\\
&\quad -\tfrac{1}{2}m_1 v_1\partial_{u_2} + \tfrac{1}{2}m_1 u_1\partial_{v_2} - \tfrac{1}{2}\lambda v_3 R_1\partial_{u_3} + \tfrac{1}{2}\lambda u_3 R_1\partial_{v_3} - \tfrac{1}{2}\lambda v_4 R_1\partial_{u_4} + \tfrac{1}{2}\lambda u_4 R_1\partial_{v_4}\\
Y^+(0,1) &= \tfrac{1}{2}m_1 v_2\partial_{u_1} - \tfrac{1}{2}m_1 u_2\partial_{v_1} + \tfrac{1}{2}[-\lambda v_2(R_3+R_4) + m_1 v_1 + 2u_{2x}]\partial_{u_2}\\
&\quad +\tfrac{1}{2}[\lambda u_2(R_3+R_4) - m_1 u_1 + 2v_{2x}]\partial_{v_2}\\
&\quad -\tfrac{1}{2}\lambda v_3 R_2\partial_{u_3} - \tfrac{1}{2}\lambda u_3 R_2\partial_{v_3} - \tfrac{1}{2}\lambda_4 v_4 R_2\partial_{u_4} + \tfrac{1}{2}\lambda u_4 R_2\partial_{v_4}\\
Y^-(0,-1) &= \tfrac{1}{2}\lambda v_1 R_3\partial_{u_1} - \tfrac{1}{2}\lambda u_1 R_3\partial_{v_1} + \tfrac{1}{2}\lambda v_2 R_3\partial_{u_2}\\
&\quad -\tfrac{1}{2}\lambda u_2 R_3\partial_{v_2} + \tfrac{1}{2}[\lambda v_3(R_1+R_2) - m_2 v_4 + 2u_{3x}]\partial_{u_3}\\
&\quad +\tfrac{1}{2}[-\lambda u_3(R_1+R_2) + m_2 u_4 + 2v_{3x}]\partial_{v_3} - \tfrac{1}{2}m_2 v_3\partial_{u_4} + \tfrac{1}{2}m_2 u_3\partial_{v_4}\\
Y^-(0,1) &= \tfrac{1}{2}\lambda v_1 R_4\partial_{u_1} - \tfrac{1}{2}\lambda u_1 R_4\partial_{v_1} + \tfrac{1}{2}\lambda v_2 R_4\partial_{u_2} - \tfrac{1}{2}\lambda u_2 R_4\partial_{v_2} + \tfrac{1}{2}m_2 v_4\partial_{u_4} - \tfrac{1}{2}m_2 u_4\partial_{v_4}\\
&\quad +\tfrac{1}{2}[\lambda v_4(R_1+R_2) + m_2 v_3 + 2u_{4x}]\partial_{u_4} + \tfrac{1}{2}[-\lambda u_4(R_1+R_2) - m_2 u_3 + 2v_{4x}]\partial_{v_4}.
\end{aligned}
\tag{2.7}
$$

The 'symmetry' in this scheme can easily be explained by the following two discrete symmetries for (2.3):

$$
\begin{aligned}
\sigma:\ & u_1 \leftrightarrows u_3 & \tau:\ & u_1 \leftrightarrows u_2\\
& v_1 \leftrightarrows v_3 & & v_1 \leftrightarrows v_2\\
& u_2 \leftrightarrows u_4 & & u_3 \leftrightarrows u_4\\
& v_2 \leftrightarrows v_4 & & v_3 \leftrightarrows v_4\\
& m_1 \leftrightarrows m_2 & & x \rightarrow -x\\
& \lambda \rightarrow -\lambda & & \lambda \rightarrow -\lambda.
\end{aligned}
\tag{2.8}
$$

Physically σ denotes the exchange of the two particles. Whenever V is a symmetry of (2.3), then $\sigma(V)$ and $\tau(V)$ are symmetries. We have

$$
\begin{aligned}
&\sigma(Y^+(i,j)) = Y^-(i,j)\\
&\tau(Y^\pm(i,j)) = Y^\pm(i,-j) \qquad \sigma^2 = \tau^2 = \mathrm{i}d.
\end{aligned}
$$

3. Non-local symmetries

A Lagrangian of (2.3) is given in [10]. By means of variational symmetries we are able to find conservation laws. Vibrational symmetries V have the property [3]:

$$pr^{(n)}V(L) + L\,\mathrm{DIV}(\xi) = 0 \tag{3.1}$$

where ξ^1 and ξ^2 are the components respectively for ∂_x and ∂_t.

Because we are looking for vertical vector fields the components ξ^1 and ξ^2 and ξ^2 are zero, and therefore the condition reduces to:

$$pr^{(n)}V(L) = 0. \tag{3.2}$$

Variational symmetries give rise to conserved currents. These can be found using Noëthers theorem. In the case of $Y^{+}(0,0)$ and $Y^{-}(0,0)$ we have the differential forms:

$$\begin{aligned} &(R_1+R_2)\,\mathrm{d}x+(-R_1+R_2)\,\mathrm{d}t \\ &(R_3+R_4)\,\mathrm{d}x+(-R_3+R_4)\,\mathrm{d}t \end{aligned} \tag{3.3}$$

with the property that the exterior derivatives are in the differential ideal described by the differential forms. These conserved currents give rise to *non-local* variables p_1 and p_2 given by

$$p_{1x}=R_1+R_2 \qquad p_{1t}=-R_1+R_2 \qquad p_{2x}=R_3+R_4 \qquad p_{2t}=-R_3+R_4. \tag{3.4}$$

Formally we can write:

$$p_1=\int_{-\infty}^{x}(R_1+R_2)\,\mathrm{d}x \qquad p_2=\int_{-\infty}^{x}(R_3+R_4)\,\mathrm{d}x. \tag{3.5}$$

In [11] new symmetries were found that include these two non-local variables. These are

$$\begin{aligned} Z^{+}(0,0)&=u_1\partial_{u_1}+v_1\partial_{v_1}+u_2\partial_{u_2}+v_2\partial_{v_2}+\lambda p_1(-v_3\partial_{u_3}+u_3\partial_{v_3}-v_4\partial_{u_4}+u_4\partial_{v_4})+2p_1\partial_{p_1} \\ Z^{-}(0,0)&=u_3\partial_{u_3}+v_3\partial_{v_3}+u_4\partial_{u_4}+v_4\partial_{v_4}+\lambda p_2(v_1\partial_{u_1}-u_1\partial_{v_1}-u_1\partial_{v_1}+v_2\partial_{u_2}-u_2\partial_{v_2})+2p_2\partial_{p_2}. \end{aligned} \tag{3.6}$$

The four symmetries $Y^{\pm}(0,\pm1)$ turned out to be divergent variational symmetries, meaning that

$$pr^{(1)}V(L)=\mathrm{DIV}(B) \tag{3.7}$$

for some B. The theorem of Noëther is applicable to these kinds of symmetries and leads to four conserved currents. Analogously to the method described above we introduce four new variables p_3, p_4, p_5 and p_6. We have

$$\begin{aligned} p_{3t}&=\tfrac{1}{2}\lambda(R_1+R_2)R_4-u_4v_{4x}+v_4u_{4x} \\ p_{3x}&=\tfrac{1}{2}\lambda(R_1+R_2)R_4+m_2(u_3u_4+v_3v_4)-u_4v_{4x}+v_4u_{4x} \\ p_{4t}&=\tfrac{1}{2}\lambda(R_1+R_2)R_3-u_3v_{3x}+v_3u_{3x} \\ p_{4x}&=\tfrac{1}{2}\lambda(R_1+R_2)R_3+m_2(u_3u_4+v_3v_4)+u_3v_{3x}-v_3u_{3x} \\ p_{5t}&=\tfrac{1}{2}\lambda(R_3+R_4)R_2+u_2v_{2x}-v_2u_{2x} \\ p_{5x}&=\tfrac{1}{2}\lambda(R_3+R_4)R_2-m_1(u_1u_2+v_1v_2)+u_2v_{2x}-v_2u_{2x} \\ p_{6t}&=-\tfrac{1}{2}\lambda(R_3+R_4)R_1-u_1v_{1x}+v_1u_x \\ p_{6x}&=\tfrac{1}{2}\lambda(R_3+R_4)R_1+m_1(u_1u_2+v_1v_2)+u_1v_{1x}-v_1u_{1x}. \end{aligned} \tag{3.8}$$

So for example

$$p_3=\int_{-\infty}^{x}\left(\tfrac{1}{2}\lambda(R_1+R_2)R_4+m_2(u_3u_4+v_3v_4)-u_4v_{4x}+v_4u_{4x}\right)\mathrm{d}x. \tag{3.8a}$$

Now we can consider the equations (2.3), (3.4) and (3.8) as a new system of differential equations which is closely related to the original system, because any solution of (2.3) immediately extends to a solution of the combined system. Our purpose is to find symmetries for this larger system of differential equations. We denote the extended system by Δ.

We expect the system Δ to have (new) first-order Lie–Bäcklund transformations, as a direct generalisation of the symmetries (3.7) found by adding non-local variables resulting from $Y^{\pm}(0,0)$.

In order to look for these transformations we introduce the exterior differential system I_Δ, describing Δ defined on the jet bundle with coordinate functions x, t, $u_1,\ldots,v_4,p_1,\ldots,p_6,u_{1x},\ldots,v_{4x}$ generated by the differential forms (2.4), (2.6) and

$$\begin{aligned}
\alpha(17)&=\mathrm{d}p_1-p_{1x}\,\mathrm{d}x-p_{1t}\,\mathrm{d}t\\
\alpha(18)&=\mathrm{d}p_2-p_{2x}\,\mathrm{d}x-p_{2t}\,\mathrm{d}t\\
\alpha(19)&=\mathrm{d}p_3-p_{3x}\,\mathrm{d}x-p_{3t}\,\mathrm{d}t\\
\alpha(20)&=\mathrm{d}p_4-p_{4x}\,\mathrm{d}x-p_{4t}\,\mathrm{d}t\\
\alpha(21)&=\mathrm{d}p_5-p_{5x}\,\mathrm{d}x-p_{5t}\,\mathrm{d}t\\
\alpha(22)&=\mathrm{d}p_6-p_{6x}\,\mathrm{d}x-p_{6t}\,\mathrm{d}t.
\end{aligned}\tag{3.9}$$

In these equations $p_{1x},p_{1t},\ldots,p_{6x},p_{6t}$ are not independent variables, but defined by (3.4) and (3.8). The exterior derivatives of $\alpha(1)$ to $\alpha(22)$ are in I_Δ by definition. The non-local variables have the following grading:

$$\begin{aligned}
&\deg(p_1)=\deg(p_2)=0 &&\deg(\partial_{p_1})=\deg(\partial_{p_2})=0\\
&\deg(p_3)=\ldots=\deg(p_6)=2 &&\deg(\partial_{p_3})=\ldots=\deg(\partial_{p_6})=-2.
\end{aligned}\tag{3.10}$$

The symmetries we are looking for are assumed to have the following properties: (i) independent of x and t; (ii) degree 2; and (iii) linear in p_3, p_4, p_5 and p_6 so that they will have the same grading as the symmetries $Y^{\pm}(0,\pm1)$. As a consequence of the assumption we know that a vector field acting as a symmetry has the form:

$$V=p_3S_3+p_4S_4+p_5S_5+p_6S_6+R \tag{3.11}$$

where $S_3,\ldots,S_6$ and R are vector fields independent of $p_3,\ldots,p_6$. Due to the fact that $\partial_{p_3},\ldots,\partial_{p_6}$ are symmetries, it is easy to see that $S_3,\ldots,s_6$ in (3.11) are Lie–Bäcklund transformations. Therefore we specify these vector fields as combinations of earlier found symmetries with degree zero. These earlier found symmetries are $Y^{\pm}(0,0)$ and $Z^{\pm}(0,0)$.

Now, we specified V in (3.11) and require (1.3) to hold with $r=1$. We obtain a system of 16 partial differential equations for the components of the vector field V, i.e.

$$I=\langle\alpha(1),\ldots,\alpha(8),\mathrm{d}\alpha(1),\ldots,\mathrm{d}\alpha(8)\rangle$$

and

$$D^1(I,p_1,\ldots,p_6)=\langle\alpha(1),\ldots,\alpha(22),\mathrm{d}\alpha(1),\ldots,\mathrm{d}\alpha(22)\rangle.$$

In these computations we made essential use of the grading in the following way. All the variables, with the exception of x, t, p_1, p_2, have a strict positive grading. For the components of V we choose to look for polynomials in these variables and the constants m_1 and m_2. The admission of only homogeneous terms in the polynomial effectively restricts the number of terms. We used this technique several times and thus cannot be sure to find all the symmetries satisfying (i) and (ii). After a huge computation we found four new symmetries. They are given by:

$$\begin{aligned}
Z^+(0,-1) &= \tfrac{1}{2}[-\lambda u_1(R_3+R_4)-m_1u_2-2v_{1x}]\partial_{u_1} \\
&\quad +\tfrac{1}{2}[-\lambda v_1(R_3+R_4)-m_1v_2+2u_{1x}]\partial_{v_1}-\tfrac{1}{2}m_1u_1\partial_{u_2}-\tfrac{1}{2}m_1v_1\partial_{v_2} \\
&\quad +\lambda p_6(v_3\partial_{u_3}-u_3\partial_{v_3}+v_4\partial_{u_4}-u_4\partial_{v_4}) \\
Z^+(0,+1) &= \tfrac{1}{2}m_1u_2\partial_{u_1}+\tfrac{1}{2}m_1v_2\partial_{v_1}+\tfrac{1}{2}[-\lambda u_2(R_3+R_4)+m_1u_1-2v_{2x}]\partial_{u_2} \\
&\quad +\tfrac{1}{2}[-\lambda v_2(R_3+R_4)+m_1v_1+2u_{2x}]\partial_{v_1} \\
&\quad +\lambda p_5(v_3\partial_{u_3}-u_3\partial_{v_3}+v_4\partial_{u_4}-u_4\partial_{v_4}) \\
Z^-(0,-1) &= -\lambda p_4(v_1\partial_{u_1}-u_1\partial_{v_1}+v_2\partial_{u_2}-u_2\partial_{v_2}) \\
&\quad +\tfrac{1}{2}[-\lambda u_3(R_1+R_2)-m_2u_4-2v_{3x}]\partial_{u_3} \\
&\quad +\tfrac{1}{2}[\lambda v_3(R_1+R_2)-m_2v_4+2u_{3x}]\partial_{v_3}-\tfrac{1}{2}m_2u_3\partial_{u_4}-\tfrac{1}{2}m_2v_3\partial_{v_4} \\
Z^-(0,+1) &= \lambda p_3(v_1\partial_{u_1}-u_1\partial_{v_1}+v_2\partial_{u_2}-u_2\partial_{v_2}) \\
&\quad +\tfrac{1}{2}m_2u_4\partial_{u_3}+\tfrac{1}{2}m_2v_4\partial_{v_3}+\tfrac{1}{2}[\lambda u_4(R_1+R_2)+m_2u_3-2v_{4x}]\partial_{u_4} \\
&\quad +\tfrac{1}{2}[\lambda v_4(R_1+R_2)+m_2v_3+2u_{4xsifi6}]\partial_{v_4}.
\end{aligned} \tag{3.12}$$

The components of $\partial_{p_1},\ldots,\partial_{p_6}$ are not calculated here, but are given in appendix I of [12].

4. The Lie algebraic structure of the non-local symmetries

In section 3 we obtained, by the introduction of non-local variables $p_3,\ldots,p_6$ associated with the conserved densities $\tilde{Y}(0,\pm 1)$ (cf [10]), four new non-local first-order Lie-Bäcklund transformations.

By prolongations of the local and non-local symmetries we are able to determine the Lie algebra structure.

Let

$$V=\phi^{u_1}\partial_{u_1}+\ldots+\phi^{v_4}\partial_{v_4} \tag{4.1}$$

be a vertical vector field. The prolongation formulae regarding the local variables are given in [3], for example

$$\phi^{u_{1x}}=D_x(\phi^{u_1}). \tag{4.2}$$

Now let p be a non-local variable, then the ∂_p components are determined from the invariance of (3.4) or (3.8). Once we have the prolongation of the vector fields we are able to compute the $\partial_{u_1},\ldots,\partial_{v_4}$ components of their Lie brackets. The components of the prolongation of the vector fields are given in appendix I of [12].

As a first result in the computations of Lie brackets of the local and non-local symmetries we obtained that each vector field denoted commutes with any vector field denoted by '−'. Morover

$$[Y^{\mp}(1,\pm 1),Z^{\pm}(0,0)]=0 \tag{4.3}$$

$$\begin{aligned}
&[Y^+(1,-1),Z^+(0,-1)]=Z^+(0,-2) \\
&[Y^+(1,-1),Z^+(0,+1)]=-\tfrac{1}{4}m_1^2Z^+(0,0) \\
&[Y^+(1,+1),Z^+(0,-1)=+\tfrac{1}{4}m_1^2Z^+(0,0) \\
&[Y^+(1,+1),Z^+(0,+1)]=Z^+(0,+2)
\end{aligned} \tag{4.4}$$

and

$$[Y^-(1,-1), Z^-(0,-1)] = Z^-(0,-2)$$
$$[Y^-(1,-1), Z^-(0,+1)] = -\tfrac{1}{4}m_2^2 Z^-(0,0)$$
$$[Y^-(1,+1), Z^-(0,-1)] = +\tfrac{1}{4}m_2^2 Z^-(0,0) \tag{4.5}$$
$$[Y^-(1,+1), Z^-(0,+1)] = Z^-(0,+2).$$

Explicit formulae for the new vector fields $Z^{\pm}(0,\pm 2)$ are given in appendix I of [12] and include non-local variables associated with the conserved functionals $\tilde{F}(Y^+_{\pm 2})$ given in [10].

Summarising these results we conclude that the action of $Y^{\pm}(1,\pm 1)$ on $Z^{\pm}(0,\pm 1)$ constitutes hierarchies of non-local symmetries of the Federbush model.

We compute the Lie bracket of $Y^+(2,0)$[10] and $Z^+(0,\pm 1)$ which results in

$$[Y^+(2,0), Z^+(0,-1)] = Z^+(1,-1)$$
$$[Y^+(2,0), Z^+(0,1)] = Z^+(1,+1) \tag{4.6}$$

where $Z^+(1,\pm 1)$ are defined by

$$Z^+(1,-1) = 2(-x+t)Z^+(0,-2) + \tfrac{1}{2}m_1^2(x+t)Z^+(0,0) + (\lambda v_1 R_{34} + m_1 v_2 - 2u_{1x})\partial_{u_1}$$
$$+ (-\lambda u_1 R_{34} - m_1 u_2 - 2v_{1x})\partial_{v_1} - \tfrac{1}{2}\lambda(v_3\partial_{u_3} - u_3\partial_{v_3} + v_4\partial_{v_4})K^+_{-1} \tag{4.7a}$$
$$Z^+(1,+1) = 2(+x+t)Z^+(0,+2) + \tfrac{1}{2}m_1^2(x-t)Z^+(0,0) + (\lambda v_2 R_{34} - m_1 v_1 - 2u_{2x})\partial_{u_2}$$
$$+ (-\lambda u_2 R_{34} - m_1 u_1 - 2v_{2x})\partial_{v_2} - \tfrac{1}{2}\lambda(v_3\partial_{u_0} - u_0\partial_{v_0} + v_4\partial_{u_4} - u_4\partial_{v_4})K^+_{+1}$$

where $K^+_{\pm 1}$ are defined by

$$K^+_{-1} = 8\int^x\int^x \tilde{F}(Y^+(0,-2)) - m_1^2\int^x\int^x \tilde{F}(Y^+(0,0))$$
$$K^+_{+1} = 8\int^x\int^x \tilde{F}(Y^+(0,+2) - m_1^2\int^x\int^x \tilde{F}(Y^+(0,0)). \tag{4.7b}$$

Equation (4.7) reflects the fact that $Y^+(2,0)$ constructs a (x,t)-dependent hierarchy $Z^+(1,*)$ from $Z^+(0,*)$ by action of the Lie bracket. Again the result is similar to the results obtained in [10] for the $Y^{\pm}(*,*)$ vector fields. So, we have no formal proof of these facts.

We end this section with the following conjectures on the existence of non-local symmetries.

(i) The non-local symmetries are ordered in a plane in a way analogous to the local symmetries $Y^{\pm}(*,*)$ (table 1).

(ii) The local symmetries $Y^{\pm}(1,\pm 1)$, $Y^{\pm}(2,0)$ act as recursion operators on the non-local symmetries in a way similar to the action in the $Y(*,*)$ plane [10].

Conclusion

From the construction of four first-order non-local symmetries we probably generate an infinite number of hierarchies of non-local symmetries $Y^{\pm}(*,*)$, created by the action of the vector fields $Y^{\pm}(1,\pm 1)$ and $Y^{\pm}(2,0)$.

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